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# Numerical Statistical Investigation on the Dynamics of Finitely Long, Nearly Periodic Chains

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**A numerical statistical analysis of the effects of structural irregularities on the dynamics of finitely long, nearly periodic assemblies is presented. In particular, emphasis is placed on the numerical analyses of the statistical behavior of the exponential decay rate of the vibration amplitude as the system is excited harmonically at one end. When the coupling is weak and the localization is strong, the localization factor (the exponential decay constant) for finitely long systems is normally distributed, with its standard deviation being inversely proportional to the square root of the distance from the excited end. Thus, for weakly coupled systems with many sites, the standard deviation of the localization factor is small compared with its mean, making the average decay factor for strong localization a true typical value. This allows us to make inferences about the population based on the information contained in the sample. For strong coupling where localization is weak, on the other hand, the probability distribution of the localization factor for finitely long systems is skewed to the right with a long tail, with its standard deviation having the same order of magnitude as that of its mean. Physically, this implies that the ensemble average for the localization factor may differ radically from the typical value. This serves as a warning against the indiscriminate use of the ensemble mean as a predictor for the behavior of a typical system when the localization is weak.**

## I. Introduction

SMALL variations in structural parameters due to manufacturing tolerances, assembly defects, or local material inhomogeneities are frequently ignored during the modeling and analyses stages of the design cycle. The rationale for such assumptions are twofold. First, many elegant solution schemes exist for perfectly periodic systems,[1–8] making their dynamic analyses computationally efficient and cost effective. Second, for most engineering structures, a perturbation in system parameters often leads only to a perturbation in the system response. Although the second assumption may often be true, it cannot be generalized to periodic structures, which consist of identical components connected to one another in an identical manner. Examples of periodic structures include truss beams on space stations, radial rib antennas, bladed disk assemblies, space reflectors, etc. Periodic structures, due to their repetitive nature, are highly sensitive to small periodicity-breaking imperfections or disorders. When certain conditions are met, small-scale local variations among the components may drastically alter the global dynamics of these periodic systems, leading to a phenomenon known as normal mode localization.[9–24]

Many solution schemes have been developed to calculate the statistics of the forced response of nearly periodic structures. Huang[25] examined the statistics of the forced response of a mistuned bladed disk assembly analytically. However, he made no attempt to characterize the probability density functions of the vibration amplitudes. Sinha[26] presented an analytical study that yields the complete information about the probability density functions of the amplitudes of a mistuned bladed disk assembly. However, no mention was made concerning the dependence of the probability density function on the disorder strength, the coupling ratio, and the number of sites. Hamade and Nikolaidis[27] presented a probabilistic vibration analysis of a two-span disordered beam with a torsional spring located at some random distance away from the middle support. They employed a second-moment method of structural reliability to estimate the probability of failure for the system. Though useful, their work and results are limited to a two-span disordered beam only. Cai and Lin[28] investigated the statistical distribution of frequency response in disordered periodic structures. They observed that the number of disordered components and the level of disorder strength can affect the standard deviation of the response. However, they did not perform a systematic study of the dependence of structural response on the disorder strength and internal coupling between components. Using the theorem on products of random matrices, Bougerol and Lacroix[29] formulated analytical means that can be used to study the statistical behavior of the localization factors for infinitely long systems. Unfortunately, most engineering structures are of finite size.

This paper presents a numerically based statistical analysis of the forced response of finitely long, nearly periodic structures. Special emphasis is placed on the derivation of the standard deviation of the localization factor $\gamma$, as well as the corresponding probability density function. In Sec. II of this paper the equations of motion are formulated, Monte Carlo wave and modal simulations are discussed, and analytical approximations of the localization factors are derived with the aid of probabilistic perturbation methods. In Sec. III, the statistical distribution of the localization factors for the limiting cases of strong and weak modal coupling are examined. In Sec. IV, the probability density functions of the localization factor are postulated for finitely long, nearly periodic chains, and these postulated results are verified by expensive Monte Carlo simulations.

The contributions of this paper are twofold. First, a complete investigation on the statistical behavior, including the mean, variance, and probability density of the localization factor for finite, nearly periodic chains, is presented. Since the localization factors are statistically distributed, to fully characterize the forced response, we must know and understand the statistics of the rate of exponential decay of the forced vibration amplitudes. Second, based on the statistical characteristics of the localization factors, we are able to make correct inferences about the behavior of a typical finitely long system.

## II. Localization Factors

### A. Equations of Motion

Consider a fixed-fixed assembly of $N$, monocoupled, one-dimensional, nearly identical component systems, each with a single degree of freedom (see Fig. 1). To study vibration transmission and wave propagation, the system is excited at its left end by a simple harmonic force of excitation frequency $\omega$, and the steady-state response at its other end is examined. Since localization mimics a damping mechanism,[30,31] no damping is included in the system to highlight the effects of irregularities on the dynamics of nearly

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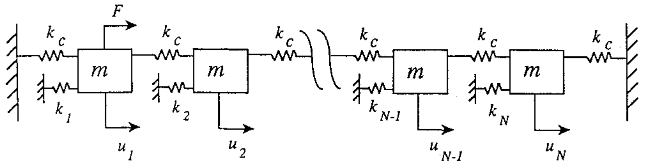


**Fig. 1 Periodic structure.**

periodic structures. For simplicity, irregularities for this system arise only in the variations among the various stiffnesses $k_i$. Thus

$$k_i = k_o(1 + \delta k_i) \tag{1}$$

where $k_o$ represents the oscillator's nominal stiffness, and $\delta k_i$ the disorder parameter for the $i$th component site. Applying Lagrange's equations, the governing equations are (see Ref. 30 for a detailed derivation)

$$\{[K] - \bar{\omega}^2[I]\}\boldsymbol{u} = \bar{\boldsymbol{F}} \tag{2}$$

where $\boldsymbol{u}$ denotes the vector of vibration amplitudes, $\bar{\boldsymbol{F}} = [\bar{F}, 0, \ldots, 0]^T$, $\bar{\omega} = \omega/\sqrt{k_o/m}$ is the dimensionless excitation frequency, $[I]$ is the identity matrix, and $[K]$ is an $N \times N$ tridiagonal matrix of the form

$$[K] = \begin{bmatrix} 1+2R+\delta k_1 & -R & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & -R & 1+2R+\delta k_i & -R & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & -R & 1+2R+\delta k_N \end{bmatrix}. \tag{3}$$

where $R$ represents a dimensionless coupling ratio, defined as $R = k_c/k_o$.

### B. Perfectly Periodic System

To better appreciate the drastic consequences of disorder in nearly periodic structures, we must understand the dynamics of an ordered periodic chain. Perfectly periodic assemblies possess the following modal properties:

1) The natural frequencies are clustered in distinct bands known as the passbands. If each substructure of the periodic assembly possesses only a single degree of freedom, the periodic chain only has one passband. Periodic systems made up of continuous elements, on the other hand, could have an infinite number of passbands.[16]

2) The modes of vibration are extended or periodic.

Perfectly periodic structures have long been known to act as mechanical filters. They have the following wave characteristics:

1) There exist alternating bands of wave attenuation and propagation.

2) In the passbands, the waves propagate (assuming no damping).

3) In the stopbands, the waves are attenuated.

Thus in the passband of an infinite, perfectly periodic chain of oscillators, the exponential decay constant is identically zero, whereas in the stopbands the exponential decay constant is large. (Since each substructure of our model has only one degree of freedom, the periodic chain of Fig. 1 consists of only one passband.) Since strong attenuation already exists in the stopbands of the nominally perfectly periodic chain, we shall focus our attention on the effects of irregularities on wave propagation and vibration transmission in the passband of the perfectly periodic system.

### C. Monte Carlo Simulations

Analytically, the localization factor $\gamma$ is obtained by letting the size of the system approach infinity.[30] Since it is impractical, not to mention uneconomical, to obtain an infinite number of components, $\gamma$ is instead calculated by taking the average over a large number of realizations $r$ of a finite disordered segment of size $N$, resulting in the localization factor $\gamma_N$.

Two simulation schemes have been proposed in the numerical analysis of the localization factor.[30,31] For strong coupling, where the localization is weak, the wave formulation is adopted, where we have an $N$-site disordered chain embedded in an otherwise ordered infinite system. (The assumption of an infinite chain allows us to obtain a closed-form expression for the localization factor. Although no engineering structure has an infinite number of sites, the dynamical characteristics of an infinite structure can frequently describe those of a finite assembly accurately.) For weak coupling, where the localization is strong, the modal formulation is recommended, where we have an $N$-site disordered assembly with fixed-fixed end conditions. The basic difference between the two approaches is that the modal formulation accounts for the boundary conditions whereas the wave approach does not. Both the wave and modal formulations are developed in Refs. 30 and 31 to allow us to shift back and forth between these two schemes, exploiting the one that is best suited to the type of confinement (weak or strong) at hand.

#### 1. *Monte Carlo Wave Simulations*

For the Monte Carlo wave simulations, the localization factor is given by

$$\gamma_N = -(1/N)\ln|\tau_N| \tag{4}$$

where $\tau_N$ denotes the transmission coefficient for the system. The transmission coefficient is simply the inverse of the (1, 1) term of the system wave transfer matrix $[W_N]$, which can be computed by multiplying the $N$ displacement transfer matrices $[T_i]$ and applying the appropriate similarity transformations.[17]

#### 2. *Monte Carlo Modal Simulations*

For the Monte Carlo modal simulations, the localization factor is given by

$$\gamma_N = -\frac{N-1}{N}\ln R + \frac{1}{N}\sum_{i=1}^{N}\ln|\bar{\omega}_i^2 - \bar{\omega}^2| \tag{5}$$

where $\bar{\omega}_i$ denotes the $i$th natural frequency of the system.[30] Although concise, the previous formulation requires the solution of an eigenvalue problem of order $N$ for every realization in the ensemble. For a large number of realizations $r$, the cost for such modal simulations becomes prohibitive, not to mention time consuming, especially if the size of the system is large. Instead of calculating $\gamma_N$ by the previous formulation, we can determine $u_N$, the response of the last site of the system, by utilizing the tridiagonal characteristics of the stiffness matrix $[K]$. Then the localization factor can be obtained from

$$\gamma_N = -(1/N)\ln|F_{1N}| \tag{6}$$

where $F_{1N}$ is the $(1, N)$ term of the $\{[K] - \bar{\omega}^2[I]\}^{-1}$ matrix, which can be easily obtained by exploiting the tridiagonality nature of the $[K]$ matrix.[31]

### D. Perturbation Methods

In general, the localization factor $\gamma$ is difficult to obtain in closed form, and we have to resort to expensive Monte Carlo simulations to calculate $\gamma$. For complicated structures, however, Monte Carlo simulations are often so computationally intensive that they become impractical. Fortunately, analytical approximations of $\gamma$ can be derived in the limiting cases of small and large disorder-to-coupling ratio with probabilistic perturbation methods.

#### 1. *Classical Perturbation Method*

In the limit of $O(\sigma/R) \ll 1$ where the localization is weak, the classical perturbation method is employed to derive an approximate analytical expression for the localization factor. (The term $O(\cdot)$ denotes the order of the argument, and $\sigma$ represents the standard deviation of the irregularities $\delta k_i$.) An infinite system is considered, such that an $N$-site disordered segment is embedded in an otherwise infinite ordered chain. In this scheme, the unperturbed system consists of the perfectly periodic assembly, whereas the perturbation consists of disorders only. Together with a wave formulation, an approximate closed-form solution can be derived for the case of strong coupling or weak localization (see Ref. 30 for the derivation):

$$\gamma^c(\bar{\omega}) = \frac{\sigma^2}{2(\bar{\omega}^2 - 1)(1 + 4R - \bar{\omega}^2)} \tag{7}$$

where the superscript $c$ denotes the classical perturbation method (CPM). From Eq. (7), we note that $\gamma^c(\bar{\omega})$ approaches infinity at $\bar{\omega}^2 = 1$ and $\bar{\omega}^2 = 1 + 4R$, which correspond to the passband-stopband edges.[32] Thus the classical approximation deteriorates near the stopbands.

#### 2. *Modified Perturbation Method*

Consider now the other limiting case of $O(\sigma/R) \gg 1$, where the localization is strong. Since the coupling $R$ is weak, we include all of the coupling terms in the perturbation matrix. To avoid multiple unperturbed eigenvalues, we leave all of the disorders in the unperturbed matrix. Assuming the irregularities to be uniformly distributed with width $2W$, an approximate closed-form solution can be derived for the case of weak coupling or strong localization (see Ref. 30 for derivation):

$$\gamma^m(\bar{\omega}) = -\ln R - 1 + \frac{1 + 2R + W - \bar{\omega}^2}{2W} \times \ln|1 + 2R + W - \bar{\omega}^2| - \frac{1 + 2R - W - \bar{\omega}^2}{2W} \ln|1 + 2R - W - \bar{\omega}^2| + \frac{1}{N} \ln R \tag{8}$$

where the superscript $m$ denotes the modified perturbation method (MPM). Equation (8) is strictly applicable to a finite disordered chain of $N$ sites, with fixed-fixed boundary conditions. The previous expression differs slightly from Eq. (44) of Ref. 30 by the last term on the right-hand side. The dependence on $N$ in Eq. (8) reflects the effects of the boundary conditions. As $N \to \infty$, the end effects vanish, and Eq. (8) becomes identical to the result given in Ref. 30.

### E. Results

The analytical results obtained via the probabilistic perturbation methods were verified with the expensive Monte Carlo simulations in Ref. 30, and excellent agreement between the two were observed. For strong coupling, the localization factor $\gamma$ is found to be maximum near the band edges and minimum near the midband. Since the localized mode shapes exhibit the same rate of decay as that of the forced response,[23,24] mistuned modes whose natural frequencies lie near the band edges are more susceptible to mode localization. For weak coupling, $\gamma$ remains nearly constant within the passband, and all mistuned modes are strongly localized.

### F. Effects of the Boundary Conditions

Figure 2 shows the midband localization factor $\gamma_{\text{mid}}$ as a function of the number of sites in the assembly, for $R = 0.01$ and $\sigma = 0.1$, where $\gamma_{\text{mid}} = \gamma(\bar{\omega} = \bar{\omega}_{\text{mid}} = \sqrt{1 + 2R})$. Also shown is $\gamma^m_{\text{mid}}$ for $N \to \infty$. Because of the boundary conditions, the agreement between the wave and modal results are poor, although they do converge as $N$ becomes large. Since the modal formulation accounts for the end effects, the traveling waves are reflected at the boundaries. This results in the reinforcement of propagating waves, leading to larger vibration amplitudes and hence smaller localization factors than the wave approach where the chain is assumed to be infinite. As expected, for $N$ large, the wave and modal solutions converge asymptotically to the modified perturbation results. Surprisingly enough, the standard deviations of the localization factors agree very well, as clearly illustrated in Fig. 3, even for values of $N$ as small as 10. The calculation of the standard deviation of the localization factor for cases where MPM is valid will be discussed in Sec III.B.

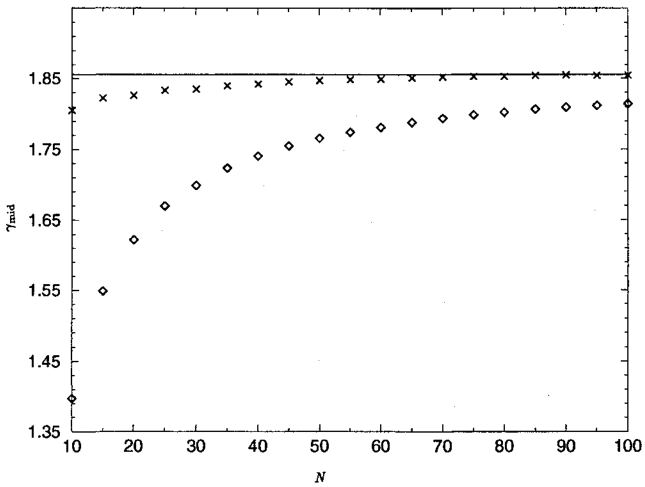


**Fig. 2 Midband localization factor, $\gamma_{\text{mid}}$, versus the number of sites, $N$, for Monte Carlo wave (x) and modal (◇) simulations. $\gamma^m_{\text{mid}}$ (superscript $m$ denotes the modified perturbation method) is also shown (—) for an infinitely long system. The system parameters are $R = 0.01$, $\sigma = 0.1$, and $r = 2{,}000$.**

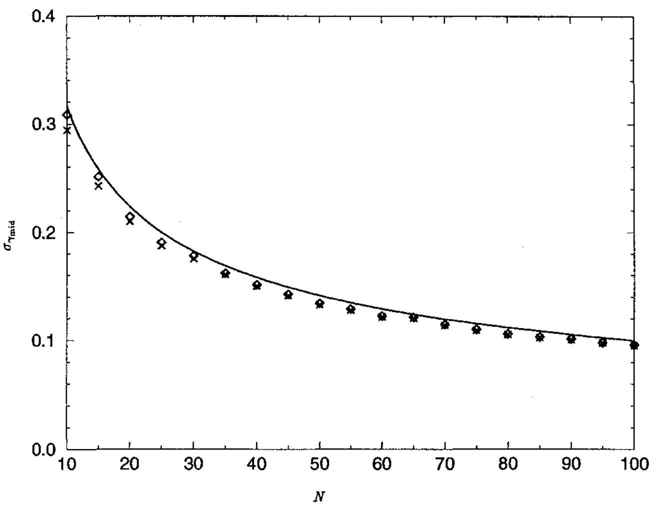


**Fig. 3 Standard deviation of midband localization factor, $\sigma_{\gamma_{\text{mid}}}$, versus the number of sites, $N$, for Monte Carlo wave (x) and modal (◇) simulations. $\sigma_{\gamma^m_{\text{mid}}}$ (superscript $m$ denotes the modified perturbation method) is also shown (—) for an infinitely long system. The system parameters are $R = 0.01$, $\sigma = 0.1$, and $r = 2{,}000$.**

## III. Statistics of the Localization Factors for Finitely Long Systems

From Monte Carlo simulations of the localization factors, it will be observed that when $\gamma$ is small (where the wave simulations are applicable), its standard deviation has the same order of magnitude as that of the mean, even for a large number of sites and realizations. Physically this implies that for small values of $\gamma$ the behavior of a typical system may deviate substantially from that of the mean response. This suggests that the probability density of $\gamma$ for the weak confinement case may exhibit some interesting behaviors.

For weak coupling where the modal simulations are valid, on the other hand, the mean of $\gamma$ is consistently one order of magnitude higher than its standard deviation. Thus for strong localization, the spread of $\gamma$ about its mean is small, indicating that for a typical realization the response is well behaved, i.e., in accordance to the ensemble mean.

For the localization effects to be significant, they must occur for typical disordered assemblies. In other words, consequences of localization are deemed meaningful if and only if the variance of $\gamma$ is much smaller than its mean. To this end, we aim to examine the standard deviation of $\gamma$ and the probability density function of $\gamma$ for finitely long systems.

### A. Strong Coupling

For $O(\sigma/R) \ll 1$, the localization factor is approximated by[30]

$$\gamma^c(\bar{\omega}) = \frac{1}{2(\bar{\omega}^2 - 1)(1 + 4R - \bar{\omega}^2)} \frac{1}{N} \sum_{l,m=1}^{N} \delta k_l \delta k_m \tag{9}$$

Assuming the disorder parameters $\delta k_l$ to be uncorrelated and identically distributed random variables, and letting the number of components $N$ approach infinity, the summation becomes

$$\lim_{N\to\infty} \frac{1}{N} \sum_{l,m=1}^{N} \delta k_l \, \delta k_m = \sigma^2 \tag{10}$$

Thus the mean localization factor reduces to

$$E[\gamma^c(\bar{\omega})] = \frac{\sigma^2}{2(\bar{\omega}^2 - 1)(1 + 4R - \bar{\omega}^2)} \tag{11}$$

which simply gives Eq. (7). The objective here is to compute the standard deviation of the localization factor as well as its mean. By definition

$$\mathrm{Var}[\gamma^c(\bar{\omega})] = E[(\gamma^c(\bar{\omega}))^2] - (E[\gamma^c(\bar{\omega})])^2 \tag{12}$$

After some lengthy algebra, the standard deviation of the localization factor is calculated to be

$$\sigma_{\gamma^c}(\bar{\omega}) = \sqrt{\mathrm{Var}[\gamma^c(\bar{\omega})]} = \frac{\sigma^2}{\sqrt{2}(\bar{\omega}^2 - 1)(1 + 4R - \bar{\omega}^2)} \tag{13}$$

Comparing Eqs. (11) and (13), we note that for this case of weak localization

$$\sigma_{\gamma^c}(\bar{\omega}) = \sqrt{2}E[\gamma^c(\bar{\omega})] = 1.414E[\gamma^c(\bar{\omega})] \tag{14}$$

Table 1 shows the simulation ratio of $\sigma_{\gamma_{\mathrm{mid}}}$ to $\gamma_{\mathrm{mid}}$ for various values of disorder strength to coupling ratios. Note the excellent agreement between the analytical and the simulated values of $\sigma_{\gamma_{\mathrm{mid}}}/\gamma_{\mathrm{mid}}$, even for a ratio of $\sigma/R$ as large as 0.2. As expected, the midband localization factor $\gamma_{\mathrm{mid}}$ increases with the ratio of disorder strength to coupling, the same trend as observed in Ref. 16. Based on the results of Table 1, we should refrain from using $\gamma$ for weak localization to infer the responses of a typical finitely long system, since the standard deviation of $\gamma$ is larger than its mean.

### B. Weak Coupling

We now turn our attention to the limiting case of $O(\sigma/R) \gg 1$. To the first order, using the modal formulation and the MPM, an approximation of the localization factor is given by[30]

$$\gamma^m(\bar{\omega}) = -\frac{N-1}{N} \ln R + \frac{1}{N} \sum_{r=1}^{N} \ln|1 + 2R - \bar{\omega}^2 + \delta k_r| \tag{15}$$

Assuming uniform distribution for the random variables, letting $N \to \infty$ only for the second term on the right-hand side (this allows a closed-form expression to be obtained), and considering the disorder parameter $\delta k_r$ as a continuous random variable $x$, the mean or expected value of the localization factor is given by

$$E[\gamma^m(\bar{\omega})] = -\frac{N-1}{N} \ln R + \frac{1}{2W} \int_{-W}^{W} \ln|1 + 2R - \bar{\omega}^2 + x| \, \mathrm{d}x \tag{16}$$

Integrating, we get

$$E[\gamma^m(\bar{\omega})] = -\ln R - 1 + \frac{1 + 2R + W - \bar{\omega}^2}{2W} \times \ln|1 + 2R + W - \bar{\omega}^2| - \frac{1 + 2R - W - \bar{\omega}^2}{2W} \ln|1 + 2R - W - \bar{\omega}^2| + \frac{1}{N} \ln R \tag{17}$$

**Table 1 Wave simulation mean and standard deviation of midband localization factor $\gamma_{\mathrm{mid}}$ for various ratios of disorder strength to coupling, $N = 20$ and $r = 2{,}000$**

| $\sigma/R$ | $\gamma_{\mathrm{mid}}$ | $\sigma_{\gamma_{\mathrm{mid}}}$ | $\sigma_{\gamma_{\mathrm{mid}}}/\gamma_{\mathrm{mid}}$ |
|---|---|---|---|
| 0.04 | 0.2029E−03 | 0.2961E−03 | 1.459 |
| 0.08 | 0.8037E−03 | 0.1142E−02 | 1.421 |
| 0.12 | 0.1784E−02 | 0.2434E−02 | 1.363 |
| 0.16 | 0.3132E−02 | 0.4052E−02 | 1.293 |
| 0.20 | 0.4844E−02 | 0.5905E−02 | 1.218 |

**Table 2 Modal simulation mean and standard deviation of midband localization factor, $\gamma_{\mathrm{mid}}$, for various ratios of disorder strength to coupling, $N = 50$ and $r = 2{,}000$**

| $\sigma/R$ | $\gamma_{\mathrm{mid}}$ | $\sigma_{\gamma_{\mathrm{mid}}}$ |
|---|---|---|
| 4.0 | 0.889 | 0.1181 |
| 8.0 | 1.547 | 0.1326 |
| 12.0 | 1.945 | 0.1362 |
| 16.0 | 2.231 | 0.1375 |
| 20.0 | 2.454 | 0.1378 |

which is the previously cited Eq. (8). Our goal is to obtain the statistics of the localization factor. For simplicity, we consider the midband excitation frequency, where $\bar{\omega}^2_{\mathrm{mid}} = 1 + 2R$. After some laborious algebra, the standard deviation of the midband localization factor, as obtained through the modified perturbation method, is calculated to be

$$\sigma_{\gamma^m_{\mathrm{mid}}} = \frac{1}{\sqrt{N}} \tag{18}$$

Note that for this case of weak coupling or strong localization the standard deviation of the midband localization factor diminishes rapidly with the number of component sites in the assembly. In fact, it can be shown that for any excitation frequency $\bar{\omega}$ the standard deviation of the localization factor is inversely proportional to the square root of the distance from the excited end, the same result that can be inferred from Scott.[33] Table 2 shows the mean and standard deviation of the localization factor at midband excitation frequency for various ratios of disorder strength to coupling. For an assembly of 50 component sites, the theoretical standard deviation is $\sigma_{\gamma^m_{\mathrm{mid}}} = 0.1414$. Note the excellent agreement between the analytical and the modal simulation results, even for a ratio of $\sigma/R$ as small as 4.

## IV. Probability Density Function of the Localization Factors for Finitely Long Systems

### A. Strong Coupling

To illustrate the probability density of the localization factors for strong coupling, consider the case of $R = 0.1$ and $\sigma = 0.01$. For simulation purposes, 20,000 realizations of 20 random sites are used. Since the ratio of disorder to coupling is small, the wave approach is chosen for Monte Carlo simulation to obtain the empirical distribution of $\gamma$. To eliminate the dependency on the excitation frequency, we focus our attention on the distribution of $\gamma$ at midband. Figure 4 displays the simulation mean and the empirical distribution of midband localization factor $\gamma_{\mathrm{mid}}$. Note that for this case of weak localization the distribution is of the exponential type. Since the mean and standard deviation are related by Eq. (14), and since the distribution appears to be exponential, we speculate that the actual continuous probability distribution is of gamma type, where the mean and standard deviation are proportional to one another. From Ref. 34, the probability density function $f(x)$ for a gamma distribution characterized by the distribution parameters $(\alpha, \beta)$ where $\beta > 0$ is given by

$$f(x) = \begin{cases} \dfrac{\beta e^{-\beta x}(\beta x)^{\alpha-1}}{(\alpha-1)!} & x \ge 0 \\ 0 & x < 0 \end{cases} \tag{19}$$

where the mean and variance of the gamma distribution are $\alpha/\beta$ and $\alpha/\beta^2$, respectively.

To confirm our conjecture, we also plot the postulated gamma distribution of the localization factor at midband. For weak localization, the postulated mean and standard deviation are obtained using Eqs. (11) and (13), respectively, with $\bar{\omega}^2 = \bar{\omega}^2_{\text{mid}} = 1 + 2R$:

$$\gamma^c_{\text{mid}} = \frac{\alpha}{\beta} = \frac{\sigma^2}{8R^2} \tag{20}$$

and the standard deviation is

$$\sigma_{\gamma^c_{\text{mid}}} = \frac{\sqrt{\alpha}}{\beta} = \frac{\sqrt{2}\sigma^2}{8R^2} \tag{21}$$

From Fig. 4, note the excellent agreement between the simulated and the postulated probability density functions. Since the distribution exhibits rightward skewness with a large tail, if we only see a single number for the expected value, we will not realize the possibility of a much higher $\gamma$ that can occur in the tail within the ensemble. Thus a typical value of $\gamma$ may deviate substantially from that of the ensemble average, indicating that the response of a typical random chain may differ considerably from that of the mean response. Hodges and Woodhouse[35,36] indicated that some ensemble averages do indeed correspond to the typical member of the ensemble, whereas others do not, since rare configurations giving very large contributions to the average may distort the answer. However, the large spread in standard deviation they observed is purely a result of the averaging scheme they employed. By simply changing from arithmetic to geometric (or logarithmic) averaging, they are able to eliminate the observed wide fluctuation about the ensemble mean. The localization factors derived in Refs. 30 and 31 are obtained by using the appropriate logarithmic averaging scheme. Mathematically it has to be so, since the amplitude decay due to localization is, on the average, exponential. Therefore the observed large fluctuation about the logarithmic mean of the localization factor for strong coupling is not due to the averaging method, but rather it is an inherent characteristic of weak localization.

Figure 4 also reveals that for this case of weak localization the greatest frequency of occurrence of the localization factor takes place at a lower decay rate than anticipated (the mean), rendering weak localization of little interest in practical engineering applications. Since the vibration amplitude is governed, on the average, by $e^{-\gamma N}$, over 690 sites are needed for the amplitude to decay by a factor of 2 when $\gamma = 0.001$. Few engineering structures, if any, consist of so many components. Furthermore, the end conditions and also that of damping may conceal the effect of disorder completely when the localization factor is small. Thus weak localization, though academically interesting, appears to be of little importance in structural dynamics. It is, however, relevant to physicists concerned with large lattices.

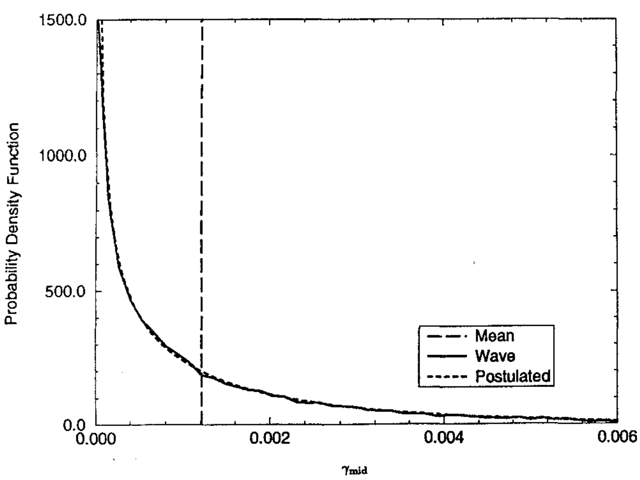


**Fig. 4 Simulated (wave) and postulated probability density functions of midband localization factor, $\gamma_{\text{mid}}$, for $R = 0.1$, $\sigma = 0.01$, $N = 20$, and $r = 20{,}000$.**

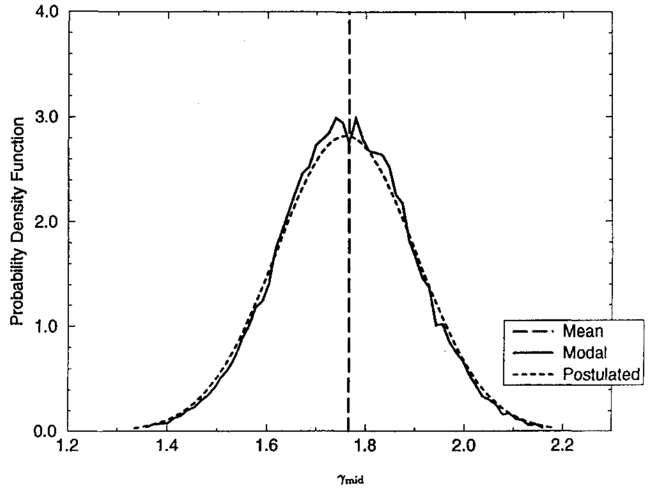


**Fig. 5 Simulated (modal) and postulated probability density functions of midband localization factor, $\gamma_{\text{mid}}$, for $R = 0.01$, $\sigma = 0.1$, $N = 50$, and $r = 20{,}000$.**

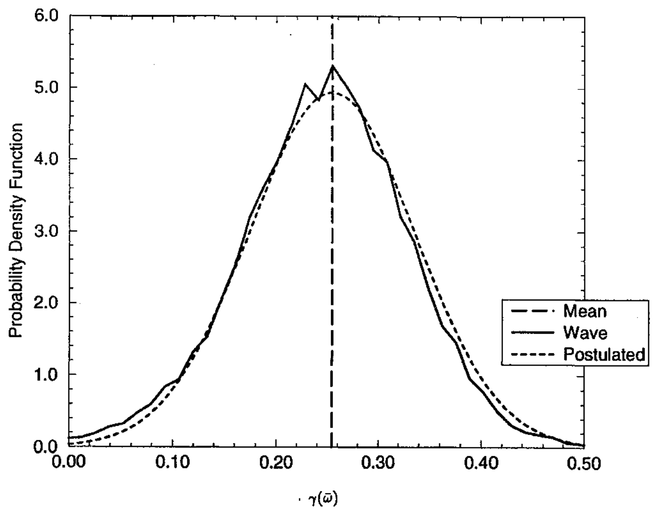


**Fig. 6 Probability density functions of localization factor at $\bar{\omega}^2 = 1 + 0.00001R$, for $R = 0.1$, $\sigma = 0.02$, $N = 20$, and $r = 20{,}000$.**

## B. Weak Coupling

Here we consider the probability density of the localization factor for the limiting case of $O(\sigma/R) \gg 1$. For simulation purposes, we consider an assembly of 50 sites and an ensemble of 20,000 realizations, for the case of $R = 0.01$ and $\sigma = 0.1$. Since the ratio of coupling to disorder is small, the Monte Carlo modal formulation is chosen for simulation. To remove the frequency dependency, we again consider midband excitation frequency. Figure 5 shows the numerically generated probability density function of midband $\gamma$. Also shown is the simulation mean of the midband localization factor. By inspection, the empirical distribution appears to be normal. From Ref. 34, the probability density function $f(x)$ for a normal distribution described by the distribution parameters $(\mu, \nu^2)$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\nu} \exp[-(x-\mu)^2/2\nu^2], \qquad -\infty < x < \infty \tag{22}$$

where the mean and standard deviation of the normal distribution are $\mu$ and $\nu$, respectively.

Figure 5 also shows the postulated probability density function at midband excitation frequency. For the case of strong localization, the mean of the postulated probability density function is given by Eq. (17), with $\bar{\omega}^2 = \bar{\omega}^2_{\text{mid}} = 1 + 2R$:

$$\gamma^m_{\text{mid}} = \mu = -\ln R + \ln\sigma + \ln\sqrt{3} - 1 + \frac{1}{N}\ln R \tag{23}$$

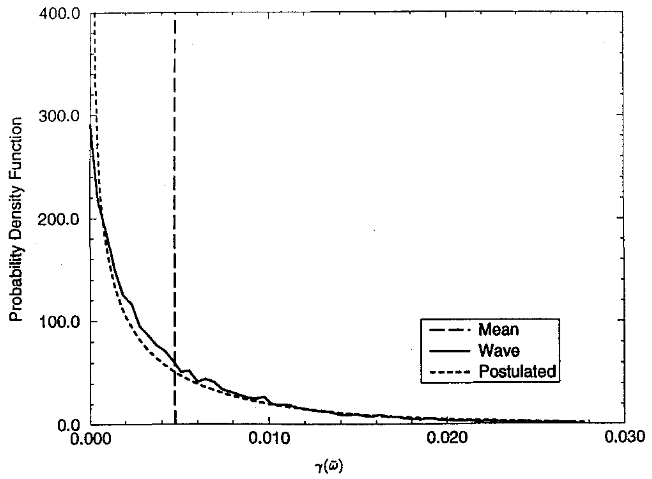


**Fig. 7 Probability density functions of localization factor at $\bar{\omega}^2 = 1+2R$ ($\gamma_{\rm mid}$), for $R = 0.1$, $\sigma = 0.02$, $N = 20$, and $r = 20{,}000$.**

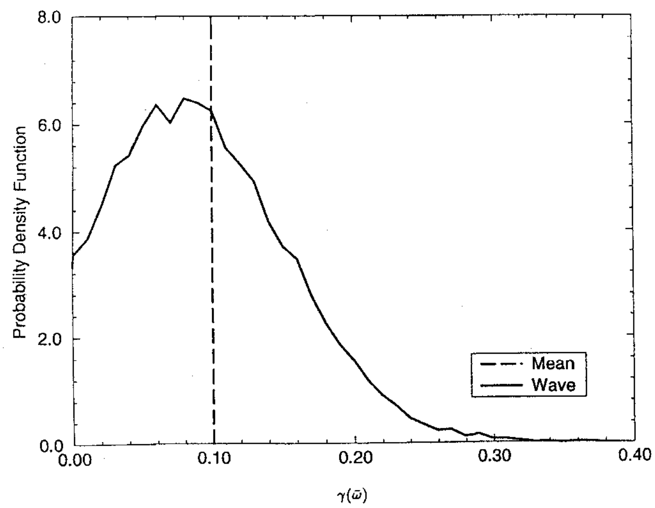


**Fig. 8 Probability density function of localization factor at $\bar{\omega}^2 = 1 + 0.01R$, for $R = 0.1$, $\sigma = 0.02$, $N = 20$, and $r = 20{,}000$.**

whereas its standard deviation is given by Eq. (18). Note the excellent agreement between the simulated and postulated density curves. For strong localization, the probability density function is normal, with the standard deviation being inversely proportional to the number of sites in the chain. Thus, as the number of components in the chain becomes large, the standard deviation rapidly becomes small compared with its mean, making the mean response a good estimator of the response of a typical system.

We have thus far considered only the probability density of the midband localization factor. We observed that for weak coupling the localization factor is normally distributed, whereas for strong coupling, the localization factor possesses a gamma distribution. For the case of strong coupling, we also noticed the strong frequency dependence of $\gamma$ on the excitation frequency, namely, $\gamma$ being large near the band edges and small near midband.[30,31] Combining the two observations, we hypothesized that the distribution of $\gamma$ varies with frequency, at least for the case of weak localization. To confirm our speculation, we investigated the probability density of $\gamma$ for $R = 0.1$ and $\sigma = 0.02$. For the system parameters chosen, since $\sigma/R < 1$, the wave formulation is selected for Monte Carlo simulations. Figures 6 and 7 show the numerical distribution of $\gamma$ for an excitation frequency near the lower band edge and also at midband, respectively. Note that for $\gamma$ near the stopband, where the localization effect is a maximum, its probability distribution is normal (Fig. 6). Since the classical approximation of Eq. (7) deteriorates near the stopbands, the parameters ($\mu$, $\nu^2$) needed to generate the postulated normal distribution are obtained numerically via Monte Carlo wave simulations. At midband excitation frequency where the localization effect is a minimum, the distribution of $\gamma$ is of gamma type (Fig. 7). Of interest is the probability density function of the localization factor at some intermediate excitation frequency. Figure 8 shows the empirical distribution at $\bar{\omega}^2 = 1 + 0.01R$. Not surprisingly, the probability density function exhibits the characteristics of both a normal and an exponential distribution. Thus within the passband in the limiting case of $O(\sigma/R) \ll 1$, the localization factor may possess different types of probability density, depending on the location of the excitation frequency within the passband spectrum. In general, for $O(\sigma/R) \ll 1$, the mean localization factors near the band edges are more representative of the behavior of a typical system, whereas the average $\gamma$ near midband cannot be used to infer the behavior of an arbitrary finite chain.

## V. Conclusion

Because of the random nature of the system parameters, the localization factors for finitely long, nearly periodic chains are shown to be statistically distributed from Monte Carlo simulations. Depending on the ratio of disorder strength to coupling, the fluctuation of the localization factor for finite, nearly periodic systems about its mean could be large. In the limiting case of small disorder strength to coupling ratio, the midband localization factor exhibits a gamma distribution, with its standard deviation being $\sqrt{2}$ times that of its expected value. Physically, this implies that the response of a typical random system may deviate dramatically from that of the mean. For a large disorder strength-to-coupling ratio, the midband localization factor is normally distributed, with its standard deviation being inversely proportional to the square root of the distance from the excited end. Thus for this limiting case of large $\sigma/R$, the dynamical behavior of any given finitely long, random periodic chain will be similar to that of the mean response. Since there could be such a large spread about the expected rate of amplitude decay depending on the ratio of disorder strength to coupling, it is of utmost importance for designer, analyst, and experimentalist to know and to understand fully the statistics of the forced response, thus allowing them to establish criteria for the existence of localization and enabling them to deal with and to design around these criteria.

Several interesting issues regarding the statistics of the localization factors remain to be investigated. First, we must explore how small damping affects the statistical behavior of the localization factor for finite, nearly periodic chains. Second, we can extend our findings to multicoupled systems where the substructure are coupled through more than one physical coordinates. Finally, we can expand the work to more elaborate systems where each substructure consists of $n$ degrees of freedom.

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